

Virtual Extension of Temperley–Lieb Algebra

Yong Zhang ^{a1}, Louis H. Kauffman ^{b2} and Mo-Lin Ge ^{a3}

^a Theoretical Physics Division, Chern Institute of Mathematics
Nankai University, Tianjin 300071, P. R. China

^b Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago, 851 South Morgan Street
Chicago, IL, 60607-7045, USA

Abstract

The virtual knot theory is a new interesting subject in the recent study of low dimensional topology. In this paper, we explore the algebraic structure underlying the virtual braid group and call it the virtual Temperley–Lieb algebra which is an extension of the Temperley–Lieb algebra by adding the group algebra of the symmetrical group. We make a connection clear between the Brauer algebra and virtual Temperley–Lieb algebra, and show the algebra generated by permutation and its partial transpose to be an example for the virtual Temperley–Lieb algebra and its important quotients.

Key Words: Temperley–Lieb Algebra, Virtual Braid Group, Brauer Algebra

¹yzhang@nankai.edu.cn

²kauffman@uic.edu

³geml@nankai.edu.cn

1 Introduction

We recall the historical development of knot theory [1] since Jones's seminal work [2, 3, 4]. A braid representation in terms of the Temperley–Lieb (TL) algebra [5, 6] leads to the formulation of the Jones polynomial, while a braid representation using the two-parameter Hecke algebra derives the HOMFLY polynomial [7]. The Birman–Wenzl algebra [8, 9] is a generalization of the skein relations of the Kauffman two-variable polynomial [10] and it maps to the Brauer algebra [11, 12] in analogy to the map of the Hecke algebra to the group algebra of the symmetric group.

In the recent study of low dimensional topology, the virtual knot theory becomes an interesting topic and this adds virtual crossings to knot theory which are a representation of the symmetric group, see [13, 14, 15, 16, 17, 18]. As a closure of classical crossings is a knot (link), a closure of classical and virtual crossings leads to a virtual knot (link). The virtual braid group is generated by classical and virtual crossings. In the present paper, we explore the algebraic structure underlying the virtual braid group and name it the virtual TL algebra which is a virtual generalization of the TL algebra by involving virtual crossings.

In the literature, the virtual TL algebra has been implicitly argued [19, 20] and independently proposed [21]. As an extension of [21], in this paper we will study the virtual TL algebra from a pure algebraic viewpoint, i.e., the mixed relations between TL idempotents and virtual crossings are determined by a presumed requirement that virtual braids can be represented in the virtual TL algebra. Hence we will be able to make the connection clear between the Brauer algebra and virtual TL algebra. The Brauer algebra [11, 12] is generated by usual TL idempotents plus an operator that behaves like a permutation, denoting an algebra of all possible connections between n points and n points in the graphical sense.

The plan of this paper is organized as follows. Section 2 derives those algebraic relations for defining the virtual Temperley–Lieb algebra and its important quotients. Section 3 builds the connection between the virtual TL algebra and the Brauer algebra, and shows that the algebra generated by permutation and its partial transpose [21] is an example for the virtual TL algebra and its important quotients. Last section remarks applications of the virtual TL algebra and virtual braid group to quantum information [22, 23].

2 The virtual TL algebra and its quotients

After the sketch of the virtual braid group and its quotients, we define the virtual TL algebra and its two important quotients: the welded virtual TL algebra and unrestricted virtual TL algebra.

2.1 The virtual braid group and its quotients

The virtual braid group VB_n [13, 14, 15, 24] is an extension of the classical braid group B_n by involving virtual crossings. Classical crossings σ_i , generators of B_n , satisfy the braid group relation called “ BGR ”,

$$\begin{aligned} BGR : \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 1, \dots, n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, & j \neq i \pm 1. \end{aligned} \quad (1)$$

Virtual crossings v_i form a representation of the group algebra of the symmetric group and satisfy the virtual crossing relation called “ VCR ”,

$$\begin{aligned} VCR : v_i^2 &= \mathbb{I}, & v_i v_{i+1} v_i &= v_{i+1} v_i v_{i+1}, \\ v_i v_j &= v_j v_i, & j &\neq i \pm 1, \end{aligned} \quad (2)$$

the symbol \mathbb{I} denoting the identity operator. Besides “ BGR ” and “ VCR ”, virtual crossings v_i and classical crossings σ_j have to satisfy the mixed relations called “ VBR ”,

$$\begin{aligned} VBR : \sigma_i v_j &= v_j \sigma_i, & j &\neq i \pm 1, \\ v_i \sigma_{i+1} v_i &= v_{i+1} \sigma_i v_{i+1}. \end{aligned} \quad (3)$$

The move with two classical crossings and one virtual crossing is a forbidden move in the virtual knot theory [24, 25, 26, 27]. There are two types of forbidden moves: the first one denoted by (F_1) and the second denoted by (F_2) ,

$$(F_1) : v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1}, \quad (F_2) : \sigma_i \sigma_{i+1} v_i = v_{i+1} \sigma_i \sigma_{i+1}. \quad (4)$$

The first forbidden move (F_1) preserves the combinatorial fundamental group, as is not true for the second forbidden move (F_2) . This makes it possible to take an important quotient of the virtual braid group VB_n . The welded braid group WB_n [24] satisfies the same isotopy relations as the VB_n group but allows the forbidden move (F_1) . The unrestricted virtual braid group UB_n allows both forbidden moves (F_1) and (F_2) although any classical knot can be unknotted in the virtual category if we allow both forbidden moves [25, 26, 27].

2.2 The virtual Temperley–Lieb algebra vTL_n

The virtual TL algebra vTL_n is a virtual extension of the TL algebra TL_n by adding virtual crossings v_i (2). The Temperley–Lieb relation called “ $TLR(\lambda)$ ”, which are satisfied by generators E_i of the TL algebra $TL_n(\lambda)$ with the loop parameter λ , are given by

$$\begin{aligned} TLR(\lambda) : E_i^2 &= \lambda E_i, & (E_i)^\dagger &= E_i, & i &= 1, \dots, n-1, \\ E_i E_{i\pm 1} E_i &= E_i, & E_i E_j &= E_j E_i, & |i-j| &> 1. \end{aligned} \quad (5)$$

Besides VCR and $TLR(\lambda)$, E_i and v_i have to satisfy the mixed relations which make it possible to represent virtual braids in the virtual Temperley–Lieb algebra.

A linear representation ρ of the VB_n group assumes a form in terms of E_i and v_i ,

$$\rho : vTL_n \rightarrow VB_n, \quad \rho_i = a + bE_i + cv_i, \quad (6)$$

where a, b, c are parameters to be determined and this representation has to satisfy the BGR and VBR relations. The VBR relation is a linear formulation of braids σ_i , and this leads to a mixed relation between E_i and v_i called “ VEV ” which is irrelevant with the parameters a, b, c ,

$$\begin{aligned} VEV : v_i E_{i+1} v_i &= v_{i+1} E_i v_{i+1}, \quad i, j = 1, 2, \dots, n-1, \\ E_i v_j &= v_j E_i, \quad j \neq i \pm 1. \end{aligned} \quad (7)$$

Note that the VEV relation together with the VCR relation derives E_{i+1} in terms of E_i, v_i, v_{i+1} by $E_{i+1} = v_i v_{i+1} E_i v_{i+1} v_i$. This means that the vTL_n algebra can be generated by the idempotent E_1 and a set of virtual crossings v_i . As a kind of heritage, the virtual braid group VB_n can be generated by the crossing σ_1 and the set of virtual crossings v_i , see [28].

With the help of the VEV , $TLR(\lambda)$ and VCR relations, the BGR relation in terms of the linear representation ρ (6) has a simplified form called the (vTL) relation,

$$\begin{aligned} (vTL) : \quad 0 &= (a^2 b + ab^2 \lambda + b^3)(E_i - E_{i+1}) + a^2 c(v_i - v_{i+1}) \\ &\quad + abc(E_i v_i + v_i E_i - E_{i+1} v_{i+1} - v_{i+1} E_{i+1}) + b^2 c \sum_{j=0}^2 [F]_j \end{aligned} \quad (8)$$

where the symbols $[F]_0$, $[F]_1$ and $[F]_2$ are given by

$$\begin{aligned} [F]_0 &= E_i v_{i+1} E_i - E_{i+1} v_i E_{i+1}, \\ [F]_1 &= v_i E_{i+1} E_i - E_{i+1} E_i v_{i+1}, \\ [F]_2 &= E_i E_{i+1} v_i - E_{i+1} E_i v_{i+1}. \end{aligned} \quad (9)$$

The (vTL) relation can be further simplified under specific circumstances. As $b = 0, ac \neq 0$, it leads to $v_i = v_{i+1}$ which is forbidden. For convenience, b is non-vanishing in the following. As $c = 0, ab \neq 0$, it is reduced to an equation of a, b , solved by setting $a = 1$,

$$a^2 b + ab^2 \lambda + b^3 = 0, \quad a = 1, \quad b_{\pm} = -\frac{1}{2}(\lambda \mp \sqrt{\lambda^2 - 4}), \quad (10)$$

which shows that $\rho_{\pm} = \mathbb{1} + b_{\pm} E$ is a braid representation. As $a = 0, bc \neq 0$, it has a simplified form,

$$b(E_i - E_{i+1}) + c \sum_{j=0}^2 [F]_j = 0. \quad (11)$$

Hence the virtual TL algebra vTL_n in the present paper is an algebra generated by TL idempotents E_i satisfying $TLR(\lambda)$ and virtual crossings v_i satisfying VCR , and E_i, v_i have to satisfy the mixed relations: VEV and (vTL) .

2.3 The welded and unrestricted virtual TL algebras

The welded and unrestricted virtual TL algebras wTL_n and uTL_n are two important quotients of the virtual TL algebra vTL_n in view of the fact that the virtual braid group VB_n has two important quotients including the welded and unrestricted virtual braid groups WB_n and UB_n .

The first forbidden move (F_1) , see (4), in terms of the braid ρ (6) has the form called the (FF_1) relation,

$$(FF_1) : a^2(v_i - v_{i+1}) = -ab(v_i E_i - E_{i+1} v_{i+1} + v_i E_{i+1} - E_i v_{i+1}) - b^2[F_1] \quad (12)$$

which is independent of the parameter c . With the help of this (FF_1) relation, the (vTL) relation is replaced by the following (wTL_1) relation,

$$(wTL_1) : \quad (a^2b + ab^2\lambda + b^3)(E_i - E_{i+1}) + b^2c([F_0] + [F_2]) + abc(E_i v_i - v_{i+1} E_{i+1} - v_i E_{i+1} + E_i v_{i+1}) = 0. \quad (13)$$

As $c = 0, ab \neq 0$, the (wTL_1) relation derives the same a, b as (10). As $a = 0, bc \neq 0$, the (wTL_1) and (FF_1) relations have the simplified forms,

$$b(E_i - E_{i+1}) + c([F]_0 + [F]_2) = 0, \quad [F]_1 = 0. \quad (14)$$

Therefore, the welded TL algebra wTL_n is generated by TL idempotents E_i satisfying $TLR(\lambda)$ and virtual crossings v_i satisfying VCR , and E_i, v_i have to satisfy the mixed relations VEV , (FF_1) and (wTL_1) .

Now we derive the (FF_2) relation from the second forbidden move (F_2) , see (4), using the braid ρ (6),

$$(FF_2) : a^2(v_i - v_{i+1}) = -ab(E_i v_i - v_{i+1} E_{i+1} + E_{i+1} v_i - v_{i+1} E_i) - b^2[F_2] \quad (15)$$

which is substituted into the (vTL) relation to derive the (wTL_2) relation,

$$(wTL_2) : \quad (a^2b + ab^2\lambda + b^3)(E_i - E_{i+1}) + b^2c([F]_0 + [F]_1) + abc(v_i E_i - E_{i+1} v_{i+1} - E_{i+1} v_i + v_{i+1} E_i) = 0. \quad (16)$$

As $c = 0, ab \neq 0$, the same a, b as (10) are required. As $a = 0, bc \neq 0$, the (FF_2) and (wTL_2) relations have the following forms,

$$b(E_i - E_{i+1}) + c([F]_0 + [F]_1) = 0, \quad [F]_2 = 0. \quad (17)$$

As a result, the unrestricted TL algebra uTL_n is generated by TL idempotents E_i satisfying $TLR(\lambda)$ and v_i satisfying VCR , and E_i, v_i have to satisfy the mixed relations VEV , (FF_l) and (wTL_l) , $l = 1, 2$. As $c = 0, ab \neq 0$, the parameters a, b are fixed by (10). As $a = 0, bc \neq 0$, the unrestricted TL algebra uTL_n satisfies the simplified mixed relations,

$$[F]_1 = [F]_2 = 0, \quad b(E_i - E_{i+1}) + c[F]_0 = 0. \quad (18)$$

3 The Brauer algebra and virtual TL algebra

We study the relationship between the virtual TL algebra and Brauer algebra, and present an example for the virtual TL algebra and its quotients which is an algebra generated by permutation and its partial transpose.

3.1 The Brauer algebra as an virtual extension of the TL algebra

The Brauer algebra $D_n(\lambda)$ [11] with the loop parameter λ is generated by TL idempotents E_i and virtual crossings v_i , $i = 1, \dots, n-1$, which satisfy the mixed relations given by

$$\begin{aligned} (ev/ve) : E_i v_i &= v_i E_i = E_i, & E_i v_j &= v_j E_i, & j &\neq i \pm 1, \\ (vee) : v_{i\pm 1} E_i E_{i\pm 1} &= v_i E_{i\pm 1}, & (eev) : E_{i\pm 1} E_i v_{i\pm 1} &= E_{i\pm 1} v_i. \end{aligned} \quad (19)$$

Obviously, the $TLR(\lambda)$, VCR and VEV relations for defining the virtual TL algebra are automatically satisfied in the Brauer algebra, and so we only need to examine under which conditions the (vTL) , (wTL_l) and (FF_l) , $l = 1, 2$ relations are satisfied in the Brauer algebra.

The three symbols F_0 , F_1 and F_2 in the (vTL) relation have the explicit forms in the Brauer algebra,

$$[F]_0 = E_i - E_{i+1}, \quad [F]_1 = v_{i+1} E_i - E_{i+1} v_i, \quad [F]_2 = E_i v_{i+1} - v_i E_{i+1}, \quad (20)$$

They are forbidden to be vanishing because $[F]_j = 0$ leads to $E_i = E_{i+1}$ which is not allowed in the Brauer algebra. The (vTL) relation is replaced by an equation in the Brauer algebra,

$$a^2 c(v_i - v_{i+1}) + b(a^2 + ab\lambda + b^2 + c(2a + b))[F]_0 + b^2 c([F]_1 + [F]_2) = 0, \quad (21)$$

which has a solution given by (10) as the objects including $v_i - v_{i+1}$, $[F]_1$, $[F]_2$ and $[F]_3$ are regarded as linearly independent. Hence *the Brauer algebra $D_n(\lambda)$ is an example for the virtual TL algebra $vTL_n(\lambda)$ as (21) can be satisfied.*

Similarly, the (wTL_1) relation for defining the welded virtual TL algebra has the form in the Brauer algebra,

$$b(a^2 + ab\lambda + b^2 + c(a + b))[F]_0 + bc(a + b)[F]_2 = 0, \quad (22)$$

which determines $a = -b$ and $\lambda = 2$ as $[F]_0$ and $[F]_2$ are linearly independent. The (FF_1) relation has a simplified form denoted by $< F_2 >$ similar to the second forbidden move (4),

$$< F_2 >: E_i^* E_{i+1}^* v_i = v_{i+1} E_i^* E_{i+1}^*, \quad E_i^* = \mathbb{1} - E_i, \quad (23)$$

where E_i^* is permutation-like $E_i^2 = \mathbb{1}$ at $\lambda = 2$ and forms a braid representation since (10). Hence *the Brauer algebra $D_n(2)$ modulo the forbidden move $< F_2 >$ is the welded virtual TL algebra $wTL_n(2)$ with $a = b = -1$.*

Furthermore, the (wTL_2) relation has a simplified form in the Brauer algebra

$$b(a^2 + ab\lambda + b^2 + c(a + b))[F]_0 + bc(a + b)[F]_2 = 0 \quad (24)$$

which specifies $a = -b$, $\lambda = 2$ and rewrites the (FF_2) relation into the form denoted by $\langle F_1 \rangle$ similar to the first forbidden move (4):

$$\langle F_1 \rangle: v_i E_{i+1}^* E_i^* = E_{i+1}^* E_i^* v_{i+1}. \quad (25)$$

Therefore, the Brauer algebra $D_n(2)$ modulo both forbidden moves $\langle F_1 \rangle$ and $\langle F_2 \rangle$ is the unrestricted virtual TL algebra $uTL_n(2)$ with $a = b = -1$.

3.2 Permutation and its partial transpose

\mathcal{H}_1 and \mathcal{H}_2 are two independent d -dimensional Hilbert spaces with bases $\{|i\rangle\}$ and $\{|j\rangle\}$, and the tensor product $|i\rangle \otimes |j\rangle$ denoted by $|ij\rangle$ are product bases of $\mathcal{H}_1 \otimes \mathcal{H}_2$. The partial transpose operator Θ_2 acts on the operator product $A \otimes B$ and only transforms indices belonging to the bases of the second Hilbert space \mathcal{H}_2 , namely $\Theta_2(A \otimes B) = A \otimes B^T$. When the bases of \mathcal{H}_2 are fixed, the symbol B^T denotes the matrix transpose.

The permutation operator P has the form by $P = \sum_{i,j=1}^d |ij\rangle \langle ji|$ which satisfies $P|\xi\eta\rangle = |\eta\xi\rangle$. With the partial transpose Θ_2 acting on the permutation P , we introduce a new operator P_* given by

$$P_* = \Theta_2 \circ P = \sum_{i,j=1}^d (|i\rangle \otimes \langle j|)(|j\rangle \otimes \langle i|)^T, \quad P_*|\xi\eta\rangle = \sum_{i=1}^d |ii\rangle \delta_{\xi\eta}. \quad (26)$$

The algebra generated by the permutation P and its partial transpose P_* is found to be the Brauer algebra $D_n(d)$. P_* is an idempotent of the $TL_n(d)$ algebra since it satisfies

$$P_* P_* = \sum_{i,j,i',j'}^d |ii\rangle \langle jj| i' i' \rangle \langle j' j' | = d \sum_{i,j}^d |ii\rangle \langle jj| = d P_*, \quad (27)$$

and the permutation P is a natural virtual crossing. The axiom (ev/ve) of the Brauer algebra is satisfied since $PP_* = P_*P = P_*$, the axiom (vee) can be checked via calculation

$$(P \otimes Id)E_2 E_1 |ijk\rangle = (P \otimes Id) \sum_{l=1}^d |kll\rangle \delta_{ij} = (Id \otimes P)E_1 |ijk\rangle, \quad (28)$$

and similarly for the verification of the axiom (eev) .

A braid representation in terms of P_* is given by

$$\rho_{\pm} = \mathbb{1} + b_{\pm} P_*, \quad b_{\pm} = -\frac{1}{2}(d \mp \sqrt{d^2 - 4}), \quad d \geq 2. \quad (29)$$

which is consistent with the case of $a = 1, c = 0, \lambda = d$ in (10). Hence P and P_* satisfy the (vTL) relation and they generate an algebra which is the virtual TL algebra $vTL_n(d)$.

At $d = 2, a = b = -1$, furthermore, P and P_* form the unrestricted virtual TL algebra $uTL_n(2)$. The operator $\mathbb{1} - P_*$ denoted by P^* has the action on $|ij\rangle$ given by

$$P^* = \mathbb{1} - P_*, \quad P^*|ij\rangle = |ij\rangle - (|00\rangle + |11\rangle)\delta_{ij}, \quad i, j = 0, 1, \quad (30)$$

and in terms of P and P^* it is easy to prove the following equations,

$$P_i P_{i+1}^* P_i^* = P_{i+1}^* P_i^* P_{i+1}, \quad P_i^* P_{i+1}^* P_i = P_{i+1} P_i^* P_{i+1}^*, \quad (31)$$

which are the forbidden moves $\langle F1 \rangle$ and $\langle F2 \rangle$ respectively.

Moreover, a diagrammatical representation for permutation and its partial transpose has been presented [21].

3.3 Concluding remarks and outlooks

In this paper, we introduce the virtual TL algebra as an extension of the TL algebra in view of the algebraic ansatz that the virtual TL algebra can represent the virtual braid group. We show that the Brauer algebra is the virtual TL algebra and its quotients under specific conditions, and present an example given by permutation and its partial transpose. In our further research, we will discuss virtual generalizations of knot invariants in terms of the trace functional over the virtual TL algebra and its quotients.

We will also study the applications of the virtual TL algebra and virtual braid group to quantum information phenomena. In [21, 22], the TL_n algebra under local unitary transformations is found to be a suitable algebraic structure underlying quantum information protocols involving maximally entangled states, and the teleportation configuration is recognized as a fundamental element in the diagrammatical representation for defining the the virtual TL algebra. The virtual mixed relation for defining the virtual braid group is a kind of formulation of the teleportation equation [22], while the virtual braid group is proposed to be a natural language for quantum computing [21].

Acknowledgements

Y. Zhang is in part supported by NSFC grants and SRF for ROCS, SEM.

For L.H. Kauffman, most of this effort was sponsored by the Defense Advanced Research Projects Agency (DARPA) and Air Force Research Laboratory, Air Force Materiel Command, USAF, under agreement F30602-01-2-05022. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright annotations thereon. The views

and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the Defense Advanced Research Projects Agency, the Air Force Research Laboratory, or the U.S. Government. (Copyright 2006.) It gives L.H. Kauffman great pleasure to acknowledge support from NSF Grant DMS-0245588.

References

- [1] L.H. Kauffman, *Knots and Physics* (World Scientific Publishers, 2002).
- [2] V.F.R. Jones, *Heck Algebra Representations of Braid Groups and Link Polynomials*, Ann. of Math. **126** (1987) 335-388.
- [3] V.F.R. Jones, *Braid Groups, Heck Algebra and Type II Factors*, Geometric Methods in Abstract Algebras, Proc. U.S.-Japan Symposium (Wiley, 1986) 242-273.
- [4] V.F.R. Jones, *A Polynomial Invariant for Knots via Von Neuman Algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985) 103-111.
- [5] H.N.V. Temperley and E.H. Lieb, *Relations between the ‘Percolation’ and ‘Colouring’ Problem and Other Graph-Theoretical Problems Associated with Regular Planar Lattices: Some Exact Results for the ‘Percolation’ Problem*, Proc. Roy. Soc. A **322** (1971) 251-280.
- [6] L.H. Kauffman and S.L. Lin, *Temperley–Lieb Recoupling Theory and Invariants of Three-Manifold*, Ann. of Math. Stud. **114** (Princeton Univ. Press, 1994).
- [7] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Miller, and A. Ocneanu, *A New Polynomial Invariant of Knots and Links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985) 239-246.
- [8] J. Birman and H. Wenzl, *Braids, Link Polynomials and a New Algebra*, Trans. Amer. Math. Soc. **313** (1989) 249-273.
- [9] J. Murakami, *The Kauffman Polynomial of Links and Representation Theory*, Osaka J. Math. **24** (1987) 745-758.
- [10] L.H. Kauffman, *An Invariant of Regular Isotopy*, Trans. Amer. Math. Monthly **95** (1988) 195-242.
- [11] R. Brauer, *On Algebras Which Are Connected With the Semisimple Continuous Groups*, Ann. of Math. **38** (1937) 857-872.
- [12] H. Wenzl, *On the Structure of Brauer’s Centralized Algebras*, Ann. of Math. **128** (1988) 179-193.

- [13] L.H. Kauffman, *Virtual Knot Theory*, European J. Comb. **20** (1999) 663-690.
- [14] L. H. Kauffman, *A Survey of Virtual Knot Theory*, in “Proceedings of Knots in Hellas 98” (World Scientific, Singapore, 2000) 143-202.
- [15] L. H. Kauffman, *Detecting Virtual Knots*, Atti. Sem. Mat. Fis. Univ. Modena, Supplemento al Vol. **IL** (2001) 241-282.
- [16] L. H. Kauffman, *A Survey of Virtual Knot Theory*, in “Proceedings of Knots in Hellas 98” (World Scientific, Singapore, 2000) 143-202.
- [17] H.A. Dye and L.H. Kauffman, *Minimal Surface Representations of Virtual Knots and Links*, Arxiv: math. GT/0401035.
- [18] V.O. Manture, *Kauffman-Like Polynomial and Curve in 2-Surfaces*, J. Knot Theory and Its Ramifications **12** (2003) 1145-1153.
- [19] H.A. Dye and L.H. Kauffman, *Virtual Knot Diagrams and the Witten-Reshetikhin-Turaev Invariant*, Arxiv: math. GT/0407407.
- [20] L.H. Kauffman, *Knot Diagrammatics*, in W. Menasco and M. Thistlethwaite (eds.), *Handbook of Knot Theory*, (Elsevier, 2005), pp. 233–318. Arxiv: math. GN/0410329.
- [21] Y. Zhang, L.H. Kauffman and R.F. Werner, *Permutation and Its Partial Transpose*. Arxiv: quant-ph/0606005. Accepted by *International Journal of Quantum Information* for publication.
- [22] Y. Zhang, *Teleportation, Braid Group and Temperley–Lieb Algebra*. J. Phys. **A: Math. Gen.** **39** (2006) 11599-11622. Arxiv: quant-ph/0610148.
- [23] Y. Zhang, *Algebraic Structures Underlying Quantum Information Protocols*. Arxiv: quant-ph/0601050.
- [24] S. Kamada, *Braid Representation of Virtual Knots and Welded Knots*, Arxiv: math. GT/0008092.
- [25] R. Fenn, R. Rimanyi, C. Rourke, *The Braid Permutation Group*, Topology **36** (1997) 123–135.
- [26] T. Kanenobu, *Forbidden Moves Unknot a Virtual Knot*, J. Knot Theory and Its Ramifications **10** (2001) 89–96.
- [27] S. Nelson, *Unknotting Virtual Knots With Gauss Diagram Forbidden Moves*, J. Knot Theory and Its Ramifications **10** (2001) 931–935.
- [28] L.H. Kauffman and S. Lambropoulou, *Virtual Braids*, Fund. Math. **184** (2004) 159-186. ArXiv: math.GT/0407349.